0.1 stepped pressure equilibrium code : readme

1. The stepped pressure equilibrium code.
2. The stepped pressure equilibrium code [1] seeks numerical solutions to macroscopic force balance between the pressure gradient and the Lorentz force in arbitrary, non-axisymmetric toroidal configurations, with fields of arbitrary topology. Generally, non-axisymmetric toroidal magnetic fields are non-integrable, so the magnetic field is not guaranteed to be tangential to a set of continually nested magnetic surfaces.
3. Equilibrium solutions are cast as extrema of a constrained energy functional.
4. Consider a plasma region comprised of a set of $N_V \equiv \texttt{Nvol}$ nested annular regions, which are separated by a discrete set of toroidal interfaces, $\mathcal{I}_l$. We insist that the fields are tangential to the interfaces. In each volume, $\mathcal{V}_l$, bounded by the $\mathcal{I}_{l-1}$ and $\mathcal{I}_l$ interfaces, the plasma energy, U_l, the global-helicity, H_l, and the mass, M_l, are given by the integrals:

   $$U_l = \int_{\mathcal{V}_l} \left(\frac{p}{\gamma-1} + \frac{B^2}{2\mu_0} \right) dv, \tag{1}$$

   $$H_l = \int_{\mathcal{V}_l} \mathbf{A}\cdot\mathbf{B}\, dv, \tag{2}$$

   $$M_l = \int_{\mathcal{V}_l} p^{1/\gamma}\, dv, \tag{3}$$

   where $\mathbf{B} = \nabla\times\mathbf{A}$. The pressure, p, is a scalar function of position.
5. The equilibrium states that we seek [2] minimize the total plasma energy, subject to the constraints of conserved helicity and conserved mass/entropy in each annular region. We allow arbitrary variations in the pressure in each annulus, δp, the magnetic field in each annulus, $\delta\mathbf{A}$, and the geometry of the interfaces, $\boldsymbol{\xi}$, except that we assume the magnetic field remains tangential to the interfaces which we consider to act as 'ideal barriers'.

   The free-energy functional we seek to extremize is

   $$F = \sum_{l=1}^{N_V} \left(U_l - \mu_l H_l/2 - \nu_l M_l \right), \tag{4}$$

   where μ_l and ν_l are Lagrange multipliers (and are constant over each volume, $\mathcal{V}_l$).
6. The first variation in the plasma energy, allowing variations in the pressure, δp, the field, $\delta\mathbf{A}$, and interface geometry, $\boldsymbol{\xi}$, is given

   $$\delta U_l = \int_{\mathcal{V}_l} \left(\frac{\delta p}{\gamma-1} + \frac{\mathbf{B}\cdot\nabla\times\delta\mathbf{A}}{\mu_0} \right) dv + \int_{\partial\mathcal{V}_l} \left(\frac{p}{\gamma-1} + \frac{B^2}{2\mu_0} \right) (\mathbf{n}\cdot\boldsymbol{\xi})\, ds, \tag{5}$$

   Using the identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$ and integrating by parts we obtain

   $$\delta U_l = \int_{\mathcal{V}_l} \left(\frac{\delta p}{\gamma-1} + \frac{\delta\mathbf{A}\cdot\nabla\times\mathbf{B}}{\mu_0} \right) dv + \int_{\partial\mathcal{V}_l} \left(\frac{p}{\gamma-1} + \frac{B^2}{2\mu_0} \right) (\mathbf{n}\cdot\boldsymbol{\xi})\, ds + \int_{\partial\mathcal{V}_l} \frac{\mathbf{n}\cdot\delta\mathbf{A}\times\mathbf{B}}{\mu_0}\, ds. \tag{6}$$

   The interfaces are assumed to be ideal, so in the surface integrals we make use of Faraday's law $\partial_t\mathbf{B} = \nabla\times\mathbf{E}$ and the ideal Ohm's law $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$ to obtain the expression $\delta\mathbf{A} = \boldsymbol{\xi}\times\mathbf{B}$. The variation in the plasma energy becomes

   $$\delta U_l = \int_{\mathcal{V}_l} \left(\frac{\delta p}{\gamma-1} + \frac{\delta\mathbf{A}\cdot\nabla\times\mathbf{B}}{\mu_0} \right) dv + \int_{\partial\mathcal{V}_l} \left(\frac{p}{\gamma-1} - \frac{B^2}{2\mu_0} \right) (\mathbf{n}\cdot\boldsymbol{\xi})\, ds. \tag{7}$$

   A similar analysis shows that the first variation in the helicity is

   $$\delta H_l = 2\int_{\mathcal{V}_l} \mathbf{B}\cdot\delta\mathbf{A}\, dv. \tag{8}$$

   The variation in the plasma mass is

   $$\delta M_l = \int_{\mathcal{V}_l} \frac{p^{1/\gamma}}{\gamma p}\delta p\, dv + \int_{\partial\mathcal{V}_l} p^{1/\gamma}(\mathbf{n}\cdot\boldsymbol{\xi})\, ds.$$

Combining these expressions, the first variation in the free-energy functional is

$$\begin{aligned}\delta F_l &= \int_{\mathcal{V}_l}\left(\frac{1}{\gamma-1}-\frac{\nu_l p^{1/\gamma}}{\gamma p}\right)\delta p\, dv \\ &+ \int_{\mathcal{V}_l}\left(\frac{\nabla\times\mathbf{B}}{\mu_0}-\mu_l\mathbf{B}\right)\cdot\delta\mathbf{A}\, dv \\ &+ \int_{\partial\mathcal{V}_l}\left(\frac{p}{\gamma-1}-\nu_l p^{1/\gamma}-\frac{B^2}{2\mu_0}\right)(\mathbf{n}\cdot\boldsymbol{\xi})\, ds\end{aligned} \tag{9}$$

The Euler-Lagrange equation for variations in the pressure is $\nu_l p^{1/\gamma} = \gamma p/(\gamma-1)$. For constant ν_l this indicates that $p = const.$ in each volume.

The Euler-Lagrange equation for variations in the variations in the vector-potential is the Beltrami equation, Eq.(10).

$$\nabla\times\mathbf{B} = \mu_l\mu_0\mathbf{B}. \tag{10}$$

The Euler-Lagrange equation for variations in the interface geometry, using $\nu_l p^{1/\gamma} = \gamma p/(\gamma-1)$, is

$$\left[\left[p + B^2/2\mu_0\right]\right] = 0. \tag{11}$$

The pressure may have discrete jumps at the interfaces, and so globally non-trivial pressure profiles may be constructed, provided the total pressure, $p + B^2/2\mu_0$, is continuous.

7. Only the variations in the geometry normal to the interfaces, $(\mathbf{n}\cdot\boldsymbol{\xi})$, are relevant: tangential variations do not alter the energy functional. To constrain the tangential degrees of freedom, additional constraints derived from minimizing the spectral width are included.

8. An auxiliary analysis [3] indicates that, in order to support non-trivial pressure, the interfaces must have strongly irrational transform.

0.1.1 numerical descretization

1. A set of N_V nested, toroidal surfaces is given on input. For expedience, we restrict attention to stellarator symmetric devices [4] so that the interfaces may be described

$$\begin{aligned}R_l(\theta,\zeta) &= \textstyle\sum_j R_{l,j}\cos(m_j\theta - n_j\zeta),\\ Z_l(\theta,\zeta) &= \textstyle\sum_j Z_{l,j}\sin(m_j\theta - n_j\zeta).\end{aligned} \tag{12}$$

2. The coordinate functions $R(s,\theta,\zeta)$ and $Z(s,\theta,\zeta)$ take the form

$$\begin{aligned}R(s,\theta,\zeta) &= \textstyle\sum_j R_j(s)\cos(m_j\theta - n_j\zeta),\\ Z(s,\theta,\zeta) &= \textstyle\sum_j Z_j(s)\sin(m_j\theta - n_j\zeta),\end{aligned} \tag{13}$$

where the functions $R_j(s)$, $Z_j(s)$ are constructed by piecewise-cubic interpolation of the $R_{l,j}$ and $Z_{l,j}$.

3. In the l-th annulus, bounded by the $(l-1)$-th and l-th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}}_l = \bar{A}_{s,l}\nabla s + \bar{A}_{\theta,l}\nabla\theta + \bar{A}_{\zeta,l}\nabla\zeta. \tag{14}$$

To this add $\nabla g_l(s,\theta,\zeta)$, where g_l satisfies

$$\begin{aligned}\partial_s g_l(s,\theta,\zeta) &= -\bar{A}_{s,l}(s,\theta,\zeta),\\ \partial_\theta g_l(s_{l-1},\theta,\zeta) &= -\bar{A}_{\theta,l}(s_{l-1},\theta,\zeta) + \psi_{t,l-1},\\ \partial_\zeta g_l(s_{l-1},0,\zeta) &= -\bar{A}_{\zeta,l}(s_{l-1},0,\zeta) + \psi_{p,l-1},\end{aligned} \tag{15}$$

for arbitrary constants $\psi_{t,l-1}$, $\psi_{p,l-1}$, which are the toroidal and poloidal-fluxes on the interior of surface $l-1$. Then $\mathbf{A}_l = \bar{\mathbf{A}}_l + \nabla g_l$ is given by $\mathbf{A}_l = A_{\theta,l}\nabla\theta + A_{\zeta,l}\nabla\zeta$ with

$$\begin{aligned}A_{\theta,l}(s_{l-1},\theta,\zeta) &= \psi_{t,l-1},\\ A_{\zeta,l}(s_{l-1},0,\zeta) &= \psi_{p,l-1}.\end{aligned} \tag{16}$$

This specifies the gauge.

4. For stellarator symmetric equilibria, $A_{\theta,l}$ and $A_{\zeta,l}$ may be represented by cosine series

$$\begin{array}{l} A_{\theta,l}(s,\theta,\zeta) = \sum_j A_{\theta,l,j}(s)\cos(m_j\theta - n_j\zeta), \\ A_{\zeta,l}(s,\theta,\zeta) = \sum_j A_{\zeta,l,j}(s)\cos(m_j\theta - n_j\zeta), \end{array} \tag{17}$$

where $A_{\theta,l,j}(s)$ and $A_{\zeta,l,j}(s)$ are represented using finite-elements.

0.1.2 compilation

1. The source is kept under CVS: `>cvs -d /u/shudson/cvs_Spec/ checkout Spec`
2. Compilation is provided by a Makefile: `>make xspec`. Try `>make help` for compilation options.
   (a) The compilation flags are given by `FLAGS`. These may be over-ruled by command line arguments.
   (b) Compilation flags must be set that convert single precision to double precision, e.g. `make FLAGS="--dbl"`.
   (c) The NAG library is used and must be correctly linked.

readme.h last modified on 2011-10-06 ;

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[1] S.R. Hudson, R.L. Dewar, M.J. Hole, and M. McGann. Nonaxisymmetric, multi-region relaxed magnetohydrodynamic equilibrium solutions. *Plasma Phys. Contr. F*, submitted, 2011.

[2] R. L. Dewar, M. J. Hole, M. Mc Gann, R. Mills, and S. R. Hudson. Relaxed plasma equilibria and entropy-related plasma self-organization principles. *Entropy*, 10:621, 2008.

[3] M. Mc Gann, R. L. Dewar, and S. R. Hudson. Hamilton-jacobi theory for continuation of magnetic field across a toroidal surface supporting a plasma pressure discontinuity. *Phys. Lett. A*, 2010.

[4] R. L. Dewar and S. R. Hudson. Stellarator symmetry. *Physica D*, 112:275, 1997.